or

$$\lambda_{v_0} = [4/(T - t_0)]\mathbf{v}_0 + [6/(T - t_0)^2]\mathbf{r}_0$$
$$\lambda_r = [6/(T - t_0)^2]\mathbf{v}_0 + [12/(T - t_0)^3]_{r_0} \quad (34)$$

From (8) and (10), these relations give the guidance law

$$\mathbf{u} = -[4/(T - t)]\mathbf{v} - [6/(T - t)^2]\mathbf{r} \quad (35)$$

where, as in (26), we have interpreted t_0, $\mathbf{v}_0$, and $\mathbf{r}_0$ as t, $\mathbf{v}$, and $\mathbf{r}$.

Using the nomenclature and assumptions of (27–32), the acceleration perpendicular to the nominal line-of-sight is given as

$$u_\perp = 4V_c\dot{\sigma} + 2V_c\sigma/(T - t) \quad (36)$$

which is a modified form of proportional navigation.

Soft Landing

For soft landing $\mathbf{a}_T = 0$, $\mathbf{a}_P = \mathbf{g}$ so that $\mathbf{a} = \mathbf{g}$, and again, as in rendezvous, we wish to make the terminal relative velocity and position very small. If we go to the limit and put $c_r \to \infty$ and $c_v \to \infty$, we obtain almost the same guidance law as (35), the only difference being the additional term $\mathbf{g}$:

$$\mathbf{u} = -\mathbf{g} - [4/(T - t)]\mathbf{v} - [6/(T - t)^2]\mathbf{r} \quad (37)$$

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Self-Contained Preliminary Orbit Determination from Angular Measurements

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Introduction

In three previous papers[1–3] the authors have established mathematical models for self-contained preliminary orbit determination (with all of the computations and measurements made on board). These models utilize measurements of the local vertical and stellar observations to determine the orientation of the orbit plane in inertial space and the increment in true anomaly between the first and succeeding measurements. The remaining two-body orbital elements result from additional measurements of altitude, radial velocity, and/or time rate of change of time anomaly. However, in practice, these latter quantities may not be obtained readily with existing onboard sensors. This difficulty is alleviated in the present paper, which extends the previous analyses to consider preliminary orbit determination from four sequential determinations of unit vectors in the radial direction, obtained via measurements of the local vertical and stellar observations and the corresponding times.

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Derivation

1. *General Remarks*

The determination of a preliminary orbit from measurement of the local vertical and stellar observations can be logically divided into three parts. First it is necessary to determine the orientation of the orbit plane and the increments in true anomaly Δv_{1j} ($j = 2, 3, 4$) between the first and succeeding measurements (here the subscripts correspond to observation times). One must then compute the in-plane orbital elements. Finally, one can determine the position and velocity vectors at the first observation time and thus fix the initial two-body orbit. The first and third steps are discussed in detail in Ref. 1 and will not be treated here. Although the requirement for four unit vectors appears to provide a redundancy in observational data, it should be noted that the only useful quantities for determining the in-plane orbital elements by this method are the increments in true anomaly. Thus, since these increments in true anomaly are obtained from the dot products of the unit vectors, four unit vectors are required to define the three increments in true anomaly necessary to determine the three in-plane orbital elements.

2. *Determination of in-plane orbital elements*

This section is concerned with the determination, from three sequential values of Δv_{1j}, of the parameter p, (semilatus rectum), orbital eccentricity e, and true anomaly at the initial observation time v_1. Two alternate formulations are presented. The first is an exact two-body determination, whereas the second is valid only for relatively low eccentricity orbits (say $e < 0.2$). The lack of generality of the second approach is compensated by considerable simplification in the formulation and a corresponding decrease in onboard computation requirements.

Exact method: The time of flight between the first and jth observations for elliptical, parabolic, and hyperbolic orbits are, respectively,[4]

$$t_j - t_1 = \left(\frac{p^3}{\mu}\right)^{1/2} \frac{1}{1 - e^2} \left\{ \frac{-e \sin v}{1 + e \cos v} + \frac{2}{(1 - e^2)^{1/2}} \times \tan^{-1}\left[\left(\frac{1 - e}{1 + e}\right)^{1/2} \tan\frac{v}{2}\right]\right\}\Bigg|_{v_1}^{v_1 + \Delta v_{1j}} \quad (1a)$$

$$t_j - t_1 = \frac{1}{2}\left(\frac{p^3}{\mu}\right)^{1/2} \tan\frac{v}{2}\left(1 + \frac{1}{3}\tan^2\frac{v}{2}\right)\Bigg|_{v_1}^{v_1 + \Delta v_{1j}} \quad (1b)$$

$$t_j - t_1 = \left(\frac{p^3}{\mu}\right)^{1/2} \frac{1}{1 - e^2} \left\{ \frac{-e \sin v}{1 + e \cos v} + \frac{2}{(e^2 - 1)^{1/2}} \tanh^{-1}\left[\left(\frac{e - 1}{e + 1}\right)^{1/2} \tan\frac{v}{2}\right]\right\}\Bigg|_{v_1}^{v_1 + \Delta v_1} \quad (1c)$$

$$j = 2, 3, 4$$

where t is the time, and μ is the product of the universal gravitational constant and the mass of the principal attracting body. It should be noted in using Eq. (1a) that the $\tan^{-1}$ term lies in the same quadrant as $v/2$. Let us define for elliptical, parabolic, and hyperbolic orbits, respectively,

$$f(e, v) = \frac{-e \sin v}{1 + e \cos v} + \frac{2}{(1 - e^2)^{1/2}} \tan^{-1} \times \left[\left(\frac{1 - e}{1 + e}\right)^{1/2} \tan\frac{v}{2}\right] \quad (2a)$$

$$f(e, v) = \tan(v/2)\left[1 + \frac{1}{3}\tan^2(v/2)\right] \quad (2b)$$

$$f(e, v) = \frac{-e \sin v}{1 + e \cos v} + \frac{2}{(e^2 - 1)^{1/2}} \tanh^{-1}\left[\left(\frac{e-1}{e+1}\right)^{1/2} \tan\frac{v}{2}\right] \tag{2c}$$

where again the $\tan^{-1}$ term in Eq. (2a) lies in the same quadrant as $v/2$. Combining Eqs. (1) and (2) yields

$$(t_m - t_1) f(e, v)\Big|_{v_1}^{v_1 + \Delta v_{12}} = (t_2 - t_1) f(e, v)\Big|_{v_1}^{v_1 + \Delta v_{1m}} \tag{3}$$

Of course, Eqs. (2a, 2b, or 2c) are selected for use in Eq. (3), depending upon whether e is less than, equal to, or greater than unity. Since Eq. (3) forms a nonlinear transcendental set, it can only be solved for v_1 and e by iteration. Having obtained v_1 and e, the semilatus rectum is given from the proper one of Eqs. (1).

Approximate method for low eccentricity orbits: The time of flight between the 1st and jth observation for any conic section orbit is given by Ref. 4:

$$t_j - t_1 = \left(\frac{p^3}{\mu}\right)^{1/2} \int_{v_1}^{v_1 + \Delta v_{1j}} \frac{dv}{(1 + e \cos v)^2} \tag{4}$$

$$j = 2, 3, 4$$

Since this section is concerned with low eccentricity orbits (say $e < 0.2$), the effects of uncertainties or large changes in the true anomaly can be minimized (by multiplication with a small number) by substituting for the eccentricity and epoch value of true anomaly the quantities

$$E_s = e \sin v_1 \tag{5a}$$

$$E_c = e \cos v_1 \tag{5b}$$

Thus, Eq. (4) can be rewritten

$$t_j - t_1 = \left(\frac{p^3}{\mu}\right)^{1/2} \int_0^{\Delta v_{1j}} \frac{d\,(\Delta v)}{[1 + E_c \cos(\Delta v) - E_s \sin(\Delta v)]^2} \tag{6}$$

$$j = 2, 3, 4$$

For the range of eccentricities considered, the denominator of the integrand of Eq. (6) can be expanded as a power series in $E_c \cos\Delta v - E_s \sin\Delta v$, yielding after term by term integration

$$\begin{aligned} t_j - t_1 = \left(\frac{p^3}{\mu}\right)^{1/2} \Big\{ & \Delta v_{1j} - 2E_c \sin\Delta v_{1j} + \\ & 2E_s (1 - \cos\Delta v_{1j}) + \frac{3}{2} E_c^2 \left(\frac{1}{2}\sin 2\Delta v_{1j} + \Delta v_{1j}\right) - \\ & 3E_cE_s \sin^2\Delta v_{1j} + \frac{3}{2} E_s^2 \left(\Delta v_{1j} - \frac{1}{2}\sin 2\Delta v_{1j}\right) - \\ & \frac{4}{3} E_c^3 \sin\Delta v_{1j}\,(2 + \cos^2\Delta v_{1j}) + 4\, E_c^2\, E_s\,(1 - \cos^3\Delta v_{1j}) - \\ & 4\, E_cE_s^2 \sin^3 \Delta v_{1j} + \frac{4}{3} E_s^3 \times \\ & [2 - \cos\Delta v_{1j}\,(2 + \sin^2\Delta v_{1j})] + \ldots \Big\} \end{aligned} \tag{7}$$

$$j = 2, 3, 4$$

where fourth- and higher-order terms in E_c, E_s and their products have been neglected. Equations (7) constitute three equations that can be solved for the three unknowns p, E_c, and E_s. It should be noted that, although the previous analysis is valid only for low eccentricity orbits, the onboard computer will not be required to evaluate transcendental inverse trigonometric functions as in the exact method. Of course, the range of acceptable eccentricities can be extended by taking additional terms in the power series expansion.

Uniqueness Proof

The proof that the iteration procedure yields one and only one value of p, e, and v_1 is as follows: Let the subscripts a and b correspond to two orbits having identical values of $v_{1j} (j = 2, 3, 4)$. Further, let us define

$$P \triangleq (p^3/\mu)^{1/4} \tag{8}$$

Then Eq. 6 can be rewritten

$$t_j - t_1 = P_a^2 \int_0^{\Delta v_{1j}} \frac{d(\Delta v)}{[1 + e_a \cos(v_{1a} + \Delta v)]^2} = P_b^2 \int_0^{\Delta v_{1j}} \frac{d(\Delta v)}{[1 + e_b \cos(v_{1b} + \Delta v)]^2} \tag{9}$$

$$j = 2, 3, 4$$

Applying the mean value theorem to Eq. (9) gives

$$\begin{aligned} 0 = \int_0^{\Delta v_{1j}} & \left\{\frac{P_a^2}{[1 + e_a \cos(v_{1a} + \Delta v)]^2} - \frac{P_b^2}{[1 + e_b \cos(v_{1b} + \Delta v)]^2}\right\} d(\Delta v) = \\ & \left\{\frac{P_a^2}{[1 + e_a \cos(v_{1a} + \epsilon_j\Delta v_{1j})]^2} - \frac{P_b^2}{[1 + e_b \cos(v_{1b} + \epsilon_j\Delta v_{1j})]^2}\right\} \Delta v_{1j} \end{aligned}$$

Therefore, except for the trivial case $\Delta v_{1j} = 0$, the preceding terms in braces must equal zero or

$$\frac{[1 + e_a \cos(v_{1a} + \epsilon_j\Delta v_{1j})]^2}{P_a^2} = \frac{[1 + e_b \cos(v_{1b} + \epsilon_j\Delta v_{1j})]^2}{P_b^2} \tag{10}$$

$$0 \le \epsilon_j \le 1 \qquad j = 2, 3, 4$$

It can be seen from the law of conservation of angular momentum that Eqs. (10) are identical to the condition that the two orbits have three unique values of time rate of change of true anomaly separated by fixed increments in true anomaly ($\epsilon_m\Delta v_{1m} - \epsilon_{m-1}\Delta v_{1m-1}$; $m = 3, 4$). Such orbits have been shown[1] to have identical values of p, e, and v_1.

Numerical Example

In order to test the method presented previously, a selenocentric satellite orbit having the following characteristics is considered: $p = 7 \times 10^6$ ft, $e = 0.1$, and $v_1 = 30°$. The increments in true anomaly between observations are taken as $\Delta v_{12} = 90°$, $\Delta v_{13} = 120°$, and $\Delta v_{14} = 150°$. Assuming that, for the moon, $\mu = 1.73140 \times 10^{14}$ ft^3/sec^2, the time increments between measurements are determined from the preceding as $t_2 - t_1 = 2.12176 \times 10^3$ sec, $t_3 - t_1 = 2.97400 \times 10^3$ sec, and $t_4 - t_1 = 3.87485 \times 10^3$ sec.

The simulated times of flight are substituted into Eq. (7). In order to initiate the iteration, the values $E_c = E_s = 0$ were selected. Two successive applications of the iteration give $p = 6.9990 \times 10^6$ ft, $e = 0.10291$, and $v_1 = 29°.423$.

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